Collapse Structures: Algebraic Foundations of Intrinsic Irreversibility

Mahmoud Manafi

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Abstract

Many mathematical and applied systems exhibit threshold behavior: beyond a certain level of accumulation, density, or complexity, structure degrades, reversibility is lost, or composition becomes impossible. Classical algebra treats such breakdowns as external anomalies. This note introduces *collapse structures* — a new algebraic framework that makes irreversibility and structural collapse intrinsic and emergent properties of the system itself. By deriving collapse from the degeneration of admissible endomorphisms under a monotone saturation invariant, the framework unifies phenomena across algebra, dynamics, geometry, topology, and beyond.

1 The Core Definition: Collapse Structure

A *collapse structure* is a triple $\mathcal{C} = (S, \sigma, \mathcal{E})$ where:

- S is a nonempty set (the state space),
- $\sigma : S \to [0, \infty]$ is a monotone saturation functional measuring structural stress, accumulation, or complexity of each state,
- $\mathcal{E} \subseteq \{f : S \to S\}$ is a collection of *admissible endomorphisms* (self-maps) satisfying

$$\sigma(f(x)) \geq \sigma(x) \quad \forall f \in \mathcal{E},\ \forall x \in S.$$

Admissible transformations never decrease saturation — they can preserve or increase it. Reversible dynamics correspond to invertible elements of $\mathcal{E}$, while irreversible ones are non-invertible.

The *critical saturation level* σ_c is defined as

$$\sigma_c := \inf\left\{\sigma(x) \ \text{ every } f \in \mathcal{E} \text{ with } \sigma(f(x)) > \sigma(x) \text{ is non-invertible}\right\}.$$

States with $\sigma(x) \geq \sigma_c$ belong to the *collapse set* C. Beyond this threshold, any further increase in saturation forces irreversibility — collapse is not postulated, but emerges structurally from the degeneration of admissible morphisms.

Algebraic operations (e.g., a partial composition $x \star y$) are then defined only in the subcritical regime ($\sigma(x) + \sigma(y) < \sigma_c$), producing a saturation-dependent algebra where axioms hold conditionally and fail structurally rather than logically.

2 Role and Significance in Mathematics

Collapse structures provide a unified algebraic language for understanding why and how systems lose coherence at thresholds — a phenomenon ubiquitous in mathematics and science:

- In dynamical systems: explains emergent irreversibility and trajectory merging without external dissipation.
- In algebraic geometry: provides an intrinsic mechanism for degeneration of morphisms and varieties.
- In statistical mechanics: models phase transitions as symmetry-breaking collapses driven by saturation accumulation.
- In computation and logic: captures unavoidable information loss as non-invertibility beyond critical complexity.
- In constructive topology and formal measure theory: enables thresholded formal covers and measures, bounding inductive constructions without choice principles.

Unlike classical structures (groups, rings, lattices) that assume global validity of properties, collapse structures treat reversibility, symmetry, and coherence as fragile and regime-dependent. Collapse is not a failure — it is the natural outcome when a system accumulates too much structural stress.

This framework reorients abstract algebra toward dynamic limits and emergent phenomena, bridging static axiomatic theories with the thresholds seen in real systems. It opens new directions in constructive mathematics, non-equilibrium modeling, and algebraic dynamics, where irreversibility becomes a derivable feature rather than an external assumption.

3 Looking Ahead

The first opportunities to work on this framework include:

- Developing categorical formulations and σ-locales (point-free extensions),
- Exploring connections to entropy, information theory, and synthetic differential geometry,
- Investigating applications to irreversible computation and threshold formal topologies.

Collapse structures invite us to rethink algebraic coherence at its breaking point — the natural next step is to apply and extend this perspective in concrete mathematical and applied contexts.